JOURNAL OF APPROXIMATION THEORY 25, 1–11 (1979)

On the Uniqueness of Best Uniform Approximations in the Presence of Constraints

WOLFGANG WARTH

Lehrstühle für Numerische und Angewandte Mathematik der Universität Gottingen, Gottingen, West Germany

Communicated by G. Meinardus

Received May 13, 1977

1. INTRODUCTION

In this paper we examine approximation problems of the following kind: Let T and E be normed spaces and P a subset of E, $F\colon E \to T$ a mapping, and let $F[P]$ be the set of approximating functions. If $w \in T$, then a best approximation to w is an element $v_0 \in F[P]$ with $\| w - v_0 \| \leqslant \| w - v \|$ for every $v \in F[P]$. We study the question whether such a best approximation is unique.

Let P be given as a set described by inequality and equality constraints: I is a finite set, $(\Gamma_j)_{j\in I}$ is a set of compact topological spaces, P_0 is an open subset of E, Z is a Banach space, and let mappings $g_{\tau,j}\colon E \to \mathbb{R}(\tau \in \Gamma_j, j \in I)$ and $p\colon E \to Z$ be given. Let

$$P = \bigcap_{j\in I} \bigcap_{\tau\in\Gamma_j} \{\alpha \in E \mid g_{\tau,j}(\alpha) \leqslant 0\} \cap \{\alpha \in E \mid p(\alpha) = \Theta\} \cap P_0 .$$

We assume here that E is a Banach space, F is Fréchet differentiable at every $\alpha \in P$, the sets $(g_{\tau,j})_{\tau\in\Gamma_j}$ $(j \in I)$ have the property D1 at every $\alpha \in P$ (see Warth [13]) and P has the property D2 at every $\alpha \in P$ (see Warth [13]).[1]

For $\alpha \in P$ let

$$I(\alpha) = \{j \in I \mid \max_{\tau\in\Gamma_j} g_{\tau,j}(\alpha) = 0\}, \qquad \Gamma_j(\alpha) = \{\tau \in \Gamma_j \mid g_{\tau,j}(\alpha) = 0\}$$

for every $j \in I$,

$$E(\alpha) = \{h \in E \mid p'(\alpha)h = \Theta, g'_{\tau,j}(\alpha)h \leqslant 0 \text{ for every } \tau \in \Gamma_j(\alpha) \text{ and } j \in I(\alpha)\}$$

and $T(\alpha) = F'(\alpha)[E(\alpha)]$. Let $d(\alpha)$ denote the dimension of span $T(\alpha)$. $\alpha \in P$ is called *regular*, if $p'(\alpha)$ is surjective and there is an $h \in E$ with $p'(\alpha)h = \Theta$ and

[1] D1 and D2 are differentiability assumptions.

0021-9045/79/010001-11$02.00/0

$g'_{\tau,j}(\alpha)h < 0$ for every $\tau \in \Gamma_j(\alpha)$ and $j \in I(\alpha)$. P is regular if every $\alpha \in P$ is regular.

In [13] it has been proved (local Kolmogoroff condition):

THEOREM 1. *If $v_0 = F(\alpha_0)$ is a best approximation to $w \in T$ and $\alpha_0 \in P$ is regular, then for every $h \in E(\alpha_0)$*

$$\min\{l \circ F'(\alpha_0)h \mid l \in \Sigma_{w-v_0}\} \leqslant 0. \tag{1.1}$$

Σ_{w-v_0} is the set of continuous, linear functionals l on T with $l(w - v_0) = \| w - v_0 \|$.

In this paper we suppose T to be the space $C(X)$ of continuous, real-valued functions defined on a compact, topological space X with the maximum norm. We present new necessary and sufficient conditions for an element to be a unique best approximation. We obtain generalizations of well-known results of Meinardus and Schwedt [9] to constrained approximation problems. We apply the general results to constrained approximation problems to obtain results as in Braess [1], Taylor [12], Loeb, Moursund, and Taylor [8], Deutsch [4], Warth [14], and Roulier and Taylor [11].

For this approximation problem (1.1) is equivalent to for every $q \in T(\alpha_0)$

$$\min\{(w(x) - v_0(x))\, q(x) \mid x \in M_{w-v_0}\} \leqslant 0, \tag{1.2}$$

where $M_{w-v_0} = \{x \in X \mid |w(x) - v_0(x)| = \| w - v_0 \|\}$. For a simple proof see Kirsch, Warth, and Werner [6].

We say that (F, P) has the *property R*, if for every $\alpha, \beta \in P$ there is a function $\phi \in C(X)$ with $\phi(x) > 0$ for every $x \in X$ and a function $q \in T(\alpha)$ such that $F(\beta) - F(\alpha) = \phi q$.

This condition has been used by Krabs [7] (where it was called "Darstellbarkeitsbedingung").

If (F, P) has property R, then (1.2) implies for every $v \in F[P]$

$$\min\{(w(x) - v_0(x))(v(x) - v_0(x)) \mid x \in M_{w-v_0}\} \leqslant 0. \tag{1.3}$$

Condition (1.3) is always sufficient for an element v_0 to be a best approximation to w (Kolmogoroff criterion). $F[P]$ is called an α-sun (see Brosowski and Wegmann [3]) if $w \in T$ and v_0 is a best approximation to w, then v_0 is a best approximation to $v_0 + \lambda(w - v_0)$ for every $\lambda > 0$. If $F[P]$ is an α-sun and v_0 is a best approximation to w, then (1.3) holds.

If (F, P) has property R, then $F[P]$ is an α-sun (best approximations are characterized by the Kolmogoroff criterion) and best approximations are characterized by the local Kolmogoroff condition.

2. A Necessary Condition

Lemma 2. *Let $\alpha_0 \in P$ be regular and let (F, P) have property* R. *Let r linear independent functions $u_1 \cdots u_r \in C(X)$ be given so that $T(\alpha_0) \subset \operatorname{span}\{u_1 \cdots u_r\}$. If there is a $\beta \in P$, $F(\beta) \neq F(\alpha_0)$ so that $F(\beta) - F(\alpha_0)$ has r zeros $x_1 \cdots x_r \in X$, then there is a function $w \in C(X)$ so that $F(\alpha_0)$ and $F(\beta)$ are best approximations to w.*

Proof. Let $G \in C(X)$ be defined by $G(x) = |v_1(x) - v_0(x)|$ $(x \in X)$, where $v_1 = F(\beta)$, $v_0 = F(\alpha_0)$. Let $\alpha = \| G \|$ and choose $x_0 \in X$ so that $G(x_0) = \alpha$. Let

$$\hat{x}_j = \begin{pmatrix} u_1(x_j) \\ \vdots \\ u_r(x_j) \end{pmatrix}, \qquad j = 0, 1, \ldots, r.$$

There is a vector $(\beta_0 \cdots \beta_r) \in \mathbb{R}^{r+1} \backslash \{\Theta\}$ so that

$$\sum_{j=0}^{r} \beta_j \hat{x}_j = \Theta. \tag{2.1}$$

If $\hat{x}_1 \cdots \hat{x}_r$ are linear independent, then $\beta_0 \neq 0$ and we can assume $\beta_0(v_1(x_0) - v_0(x_0)) > 0$.

If $\hat{x}_1 \cdots \hat{x}_r$ are linear dependent we can assume $\beta_0 = 0$. Let

$$J = \{j \in \{0 \cdots r\} \mid \beta_j \neq 0\}.$$

There is a function $g \in C(X)$, $\| g \| = 1$ so that

$$g(x_j) = \frac{\beta_j}{|\beta_j|}, \qquad j \in J.$$

Let $w \in C(X)$ be defined by

$$w(x) = g(x)(\alpha - G(X)) + v_1(x) \qquad (x \in X).$$

Then $\| w - v_0 \| = \alpha$. For $j \in J \backslash \{0\}$ we obtain

$$w(x_j) - v_0(x_j) = \frac{\beta_j}{|\beta_j|} \alpha$$

hence $\{t_j \mid j \in J\} \subset M_{w-v_0}$.

(2.1) implies $\sum_{j=0}^{r} \beta_j h(x_j) = 0$ for every $h \in \operatorname{span}\{u_1 \cdots u_r\}$, particularly for every $h \in T(\alpha_0)$. Thus

$$\min\{\beta_j h(x_j) \mid j \in J\} \leqslant 0$$

for every $h \in T(\alpha_0)$.

For $j \in J \backslash \{0\}$ $\beta_j(w(x_j) - v_0(x_j)) > 0$ and if $0 \in J$ then $\beta_0(w(x_0) - v_0(x_0)) > 0$. Consequently

$$\min\{(w(x_j) - v_0(x_j))\, h(x_j) \mid j \in J\} \leqslant 0$$

for every $h \in T(\alpha_0)$. Since (F, P) has property R, v_0 is a best approximation to w by the Kolmogoroff criterion. Since $\| w - v_1 \| \leqslant \alpha$, v_1 is a best approximation as well.

(F, P) has the *global Haar property*, if

— $d(\alpha) < \infty$ for every $\alpha \in P$,

— for every $v \in F[P]$ there is an $\alpha \in P$ so that $F(\alpha) = v$ and $d(\alpha) \geqslant 1$,

— $v, \tilde{v} \in F[P]$ and $v \neq \tilde{v}$ imply that $v - \tilde{v}$ has at most $d_{\tilde{v}} - 1$ zeros with

$$d_{\tilde{v}} = \min\{d(\alpha) \mid \alpha \in P, F(\alpha) = \tilde{v}, d(\alpha) \geqslant 1\}.$$

Applying Lemma 2 we obtain

THEOREM 3. *Let P be regular and let (F, P) have property R. If every $w \in C(X)$ has at most one best approximation and $1 \leqslant d(\alpha) < \infty$ for every $\alpha \in P$, then (F, P) has the global Haar property.*

Proof. Suppose there are $v, \tilde{v} \in F[P]$ so that $v - \tilde{v}$ has $d_{\tilde{v}} - 1$ zeros. Let $r = d_{\tilde{v}}$, $\tilde{\alpha} \in P$ so that $F(\tilde{\alpha}) = \tilde{v}$ and $d(\tilde{\alpha}) = r$. Choose $u_1, \ldots, u_r \in C(X)$ so that $T(\tilde{\alpha}) \subset \operatorname{span}\{u_1 \cdots u_r\}$. Then there is a function $w \in C(X)$ so that v and $\tilde{v}$ are best approximations to w according to Lemma 2 and we obtain a contradiction.

EXAMPLE. Let $E = \mathbb{R}^n$, $u_1 \cdots u_n \in C(X)$ linear independent. Let F: $E \to C(X)$ be defined by $(\alpha_1 \cdots \alpha_n) \mapsto \sum_{i=1}^n \alpha_i u_i$. $I = \{1, 2, \ldots, n\}$. Let g_j: $E \to \mathbb{R}$ $(j \in J)$ be defined by $(\alpha_1 \cdots \alpha_n) \mapsto -\alpha_j$. Then $P = \{(\alpha_1 \cdots \alpha_n) \in \mathbb{R}^n \mid \alpha_j \geqslant 0\ j = 1, 2, \ldots, n\}$ is regular. For $\alpha \in P$ let

$$I(\alpha) = \{j \in I \mid \alpha_j = 0\},$$

$$E(\alpha) = \{(y_1 \cdots y_n) \in \mathbb{R}^n \mid y_j \geqslant 0 \text{ for every } j \in I(\alpha)\},$$

$$T(\alpha) = \left\{ \sum_{j=1}^n y_j u_j \in C(X) \,\middle|\, y_i \geqslant 0 \text{ for every } j \in I(\alpha) \right\},$$

$$d(\alpha) = n.$$

Since (F, P) has property R and P is regular we obtain the implication:

If every $w \in C(X)$ has at most one best approximation then $\operatorname{span}\{u_1 \cdots u_n\}$ is a Haar space. (Apply Theorem 3). But the inverse implication does not hold. Let $X = [-1, 1]$, $n = 3$, and $u_1(x) = 1$, $u_2(x) = x$, $u_3(x) = x^2$ for every $x \in X$.

Let $w(x) = x^2 - 1$ for every $x \in X$. Then Θ and x^2 are best approximations to w.

3. A Sufficient Condition

A *T-signature* (M^+, M^-) is a pair of closed, disjoint G_δ-subsets of X with $M^+ \cup M^- \neq \varnothing$. Let $\hat{M} = (M^+, M^-)$ and

$$\epsilon_{\hat{M}}(x) = 1 \qquad \text{if} \quad x \in M^+,$$
$$\epsilon_{\hat{M}}(x) = -1 \qquad \text{if} \quad x \in M^-.$$

The T-signature $\hat{M}$ is called *extremal* for $v_0 \in F[P]$ if for every $v \in F[P]$, $\min\{\epsilon_{\hat{M}}(x)(v(x) - v_0(x)) \mid x \in M^+ \cup M^-\} \leqslant 0$. If $w \in C(X)$, $w \neq \Theta$ then the T-signature $\hat{M}_w = (M_w{}^+, M_w{}^-)$ is defined by $M_w{}^+ = \{x \in X \mid w(x) = \| w \|\}$, $M_w{}^- = \{x \in X \mid w(x) = -\| w \|\}$.

Lemma 4. *Let $\alpha_0 \in P$ be regular. If $\hat{M}$ is a T-signature which is extremal for $v_0 = F(\alpha_0)$, then for every $q \in T(\alpha_0)$, $\min\{\epsilon_{\hat{M}}(x)\, q(x) \mid x \in M^+ \cup M^-\} \leqslant 0$.*

Proof. There is a function $\epsilon \in C(X)$ with $M_\epsilon{}^+ = M^+$, $M_\epsilon{}^- = M^-$, and $\epsilon(x) = \epsilon_{\hat{M}}(x)$ for every $x \in M^+ \cup M^-$. By the Kolmogoroff criterion v_0 is a best approximation to $v_0 + \epsilon$ since $\hat{M}$ is extremal for v_0. Hence by Theorem 1 we obtain the result.

The set of zeros of a function $w \in C(X)$ is denoted by $Z(w)$. If $\hat{M}$ and $\hat{M}_1$ are two T-signatures and $M_1 \subset M^+$, $M_1 \subset M^-$ then we write $\hat{M}_1 < M$.

Using an idea of Brosowski and Wegmann [3, "Durchschnittssatz"] we obtain:

Lemma 5. *Let $F[P]$ be an α-sun. Let $w \in C(X)$ and suppose $v_0, v_1 \in F[P]$ are best approximations to w. Then $\hat{M} = (M^+, M^-)$ with*

$$M^+ = M^+_{w-v_0} \cap M^+_{w-v_1}, \qquad M^- = M^-_{w-v_0} \cap M^-_{w-v_0}$$

is a T-signature which is extremal for v_0 (and $Z(v_1 - v_0) \supset M^+ \cup M^-$).

Proof. Let $\lambda > 0$ and $w_\lambda = w + \lambda(w - v_1)$. Since $F[P]$ is an α-sun v_1 is a best approximation to w_λ and $\| w_\lambda - v_1 \| = (1 + \lambda) \| w - v_1 \|$. However $\| w_\lambda - v_0 \| \leqslant \| w_\lambda - w \| + \| w - v_0 \| \leqslant \lambda \| w - v_1 \| + \| w - v_0 \| = (1 + \lambda)$ $\| w - v_1 \|$. Hence v_0 is a best approximation to w_λ.

Since $F[P]$ is an α-sun

$$\min_{x \in M_{w_\lambda - v_0}} (w_\lambda(x) - v_0(x))(v(x) - v_0(x)) \leqslant 0$$

for every $v \in F[P]$. (The Kolmogoroff criterion is necessary.)

$$M^+_{w_\lambda - v_0} = \{x \in X \mid w(x) - v_0(x) + \lambda(w(x) - v_1(x)) = (1 + \lambda) \| w - v_1 \|\}$$
$$= M^+_{w-v_0} \cap M^+_{w-v_1} = M^+$$

and

$$M^-_{w_\lambda - v_0} = M^-.$$

Hence $\min_{x \in M^+ \cup M^-} \epsilon_M(x)(v(x) - v_0(x)) \leqslant 0$ for every $v \in F[P]$, i.e., M is extremal for v_0.

Let $v_0 \in F[P]$ and $\hat{M}$ is a T-signature. (F, P) has the *property TU* at $(v_0, \hat{M})$ if the following holds:

If $\hat{M}_1 < \hat{M}$ and if there is a $v \in F[P]$, $v \neq v_0$ with $Z(v - v_0) \supset M_1^+ \cup M_1^-$, then there is an $\alpha \in P$ so that

(i) $F(\alpha) \neq v_0$

(ii) $Z(F(\alpha) - v_0) \supset M_1^+ \cup M_1^-$

(iii) there is an $q \in T(\alpha)$ so that $\epsilon_{\hat{M}_1}(x)\, q(x) > 0$ for every $x \in M_1^+ \cup M_1^-$.

(F, P) has the property TU if it has the property at $(v_0, \hat{M})$ for every $v_0 \in F[P]$ and every T-signature $\hat{M}$.

THEOREM 6. *If P is regular, $F[P]$ is an α-sun, $w \in C(X)$, and $v_0 \in F[P]$ is a best approximation to w, then v_0 is the only best approximation if (F, P) has the property TU at $(v_0, \hat{M}_{w-v_0})$.*

Proof. Suppose there is a $v \in F[P]$, $v \neq v_0$ which is a best approximation to w. By Lemma 5 $M^+ = M^+_{w-v_0} \cap M^+_{w-v}$, $M^- = M^-_{w-v_0} \cap M^-_{w-v}$ defines a T-signature $\hat{M} = (M^+, M^-)$ with $\hat{M} < \hat{M}_{w-v_0}$ and $Z(v - v_0) \supset M^+ \cup M^-$. Then there is an $\alpha \in P$ with $F(\alpha) \neq v_0$, $Z(F(\alpha) - v_0) \supset M^+ \cup M^-$ and there is a $q_0 \in T(\alpha)$ with $\epsilon_{\hat{M}}(x)\, q_0(x) > 0$ for every $x \in M^+ \cup M^-$. $\hat{M}$ is extremal for v_0. $Z(F(\alpha) - v_0) \supset M^+ \cup M^-$ implies that $\hat{M}$ is extremal for $F(\alpha)$. Then Lemma 4 implies $\min\{\epsilon_{\hat{M}}(x)\, q(x) \mid x \in M^+ \cup M^-\} \leqslant 0$ for every $q \in T(\alpha)$ and we obtain a contradiction with q_0.

As a corollary we obtain

THEOREM 7. *If P is regular, $F[P]$ is an α-sun and (F, P) has the property TU, then every $w \in C(X)$ has at most one best approximation.*

Examples

(1) If (F, P) has the global Haar property and $T(\alpha)$ is a Haar space on X for every $\alpha \in P$,[2] then (F, P) has the property TU. Applying Theorem 7

[2] A finite dimensional linear subspace $H \neq \{\Theta\}$ of $C(X)$ is called a Haar space on X, if every $v \in H$ has at most dim $H - 1$ zeros (in X).

we obtain the generalization of a well-known theorem of Meinardus and Schwedt [9, Satz 14].

(2) If (F, P) has property R and $T(\alpha)$ is a Haar space on X for every $\alpha \in P$, then (F, P) has the global Haar property. Applying Theorem 7 (and a theorem of Krabs [7] on the relation of the property R to asymptotic conbexity) we obtain a result similar to Satz 10 of Meinardus and Schwedt [9].

If $P = E$, $n \in \mathbb{N}$, $u_1 \cdots u_n \in C(X)$ are linear independent and $F: E \to C(X)$ is given by $(\alpha_1 \cdots \alpha_n) \mapsto \Sigma\alpha_i u_i$, then Theorems 3 and 7 imply the well-known theorem (see [10]):

Every $w \in C(X)$ has at most one best approximation if and only if $\text{span}\{u_1 \cdots u_n\}$ is a Haar space on X.

Brosowski [2] claimed the equivalence of the uniqueness of best approximations and the space $T(\alpha)$ to be Haar spaces on X in the case of asymptotic convexity. However in the proof there is a gap.

(3) Suppose that for every $v_0 \in F[P]$ there is a $\alpha_0 \in P$ with $F(\alpha_0) = v_0$, $T(\alpha_0)$ is a Haar space on X, and every difference $v - v_0$, $v \in F[P]$, $v \neq v_0$ has at most $d(\alpha_0) - 1$ zeros, then (F, P) has property TU.

In the following three examples let $n \in \mathbb{N}$, $P_0 = E = \mathbb{R}^n$, $X = [0, 1]$ and let $F: E \to C[0, 1]$ be given by $(\alpha_1 \cdots \alpha_n) \mapsto \Sigma\alpha_i u_i$, where $u_1 \cdots u_n \in C[0, 1]$ are linear independent.

(4) *Linear Approximation with Parameter Constraints*

Let $m \in \mathbb{N}$, $m \leqslant n$. Suppose for every set I with $\{1, 2,..., m\} \subset I \subset \{1, 2,..., n\}$ $\text{span}\{u_i \mid i \in I\}$ is a Haar space on X. Let $P = \{(\alpha_1 \cdots \alpha_n) \in E \mid \alpha_j \geqslant 0\, j = m + 1 \cdots n\}$. Then P is regular and (F, P) has the property R.

Let $v_0 \in F[P]$, $\hat{M}$ is a T-signature. Suppose there is a signature $\hat{M}_1 < \hat{M}$ and a $v_1 \in F[P]$, $v_1 \neq v_0$ so that $Z(v_1 - v_0) \supset M_1^+ \cup M_1^-$; $\hat{v} = \Sigma\hat{\alpha}_j u_j = \frac{1}{2}(v_1 + v_0)$. Let r be the number of indices $j \in \{m + 1 \cdots n\}$ with $\hat{\alpha}_j > 0$ and $I = \{j \in \{m + 1 \cdots n\} \mid \hat{\alpha}_j > 0\}$. Then $\hat{v} - v_0 \in \text{span}(\{u_1 \cdots u_m\} \cup \{(u_i)_{i \in I}\})$ and $\hat{v} - v_0$ has at most $m + r - 1$ zeros. Then there is a $q \in T(\hat{\alpha})$ with $\epsilon_{\hat{M}_1}(x)\, q(x) > 0$ for every $x \in M_1^+ \cup M_1^-$. Hence (F, P) has property TU at $(v_0, \hat{M})$ (for every $v_0 \in F[P]$ and every T-signature $\hat{M}$).

(5) *Linear Restricted Range Approximation*

Suppose $U = \text{span}\{u_1 \cdots u_n\}$ is a Haar space on X. $I = \{1, 2\}$, $T = [0, 1]$. Let $l, u \in C[0, 1]$ so that $l(x) < u(x)$ for every $x \in [0, 1]$. $g_{\tau,1}: E \to \mathbb{R}$ $(\tau \in \Gamma)$ is defined by

$$(\alpha_1 \cdots \alpha_n) \mapsto \Sigma\alpha_i u_i(\tau) - u(\tau)$$

and $g_{\tau,2}: E \to \mathbb{R}$ $(\tau \in \Gamma)$ by

$$(\alpha_1 \cdots \alpha_n) \mapsto l(\tau) - \Sigma \alpha_i u_i(\tau).$$

Then

$$P = \{(\alpha_1 \cdots \alpha_n) \in E \mid g_{\tau,1}(\alpha_1 \cdots \alpha_n) \leqslant 0, \qquad g_{\tau,2}(\alpha_1 \cdots \alpha_2) \leqslant 0 \text{ for every } \tau \in \Gamma\}$$

is regular as was shown by Warth [13] and (F, P) has property R.

If $w \in C[0, 1]$ and $v_0 \in F[P]$ so that

$$(M^+_{w-v_0} \cap \{x \in [0, 1] \mid v_0(x) = u(x)\}) \cup (M^-_{w-v_0} \cap \{x \in [0, 1] \mid v_0(x) = l(x)\}) = \varnothing$$

then (F, P) has the property TU at $(v_0, \hat{M}_{w-v_0})$. For $\hat{M} < \hat{M}_{w-v_0}$ and $v \in F[P]$, $v \neq v_0$ with $Z(v - v_0) \supset M^+ \cup M^-$. $A = \{x \in [0, 1] \mid \hat{v}(x) = u(x)\}$, $B = \{x \in [0, 1] \mid \hat{v}(x) = l(x)\}$, where $\hat{v} = \frac{1}{2}(v + v_0) = \Sigma \hat{\alpha}_j u_j$. Then

$$T(\hat{\alpha}_1 \cdots \hat{\alpha}_n) = \{q \in U \mid q(x) \leqslant 0 \text{ for every } x \in A, \qquad q(x) \geqslant 0 \text{ for every } x \in B\}.$$

$Z(\hat{v} - v_0) = Z(v - v_0) \supset M^+ \cup M^-$ implies $Z(\hat{v} - v_0) \supset M^+ \cup M^- \cup A \cup B$. Since U is a Haar space on $[0, 1]$ $M^+ \cup M^- \cup A \cup B$ contains at most $n - 1$ points. $(A \cap M^+) \cup (B \cap M^-) = \varnothing$ implies that there is a $q \in U$ so that $q(x) = 1$ if $x \in M^+$, $q(x) = -1$ if $x \in M^-$, $q(x) \leqslant 0$ if $x \in A$ and $q(x) \geqslant 0$ if $x \in B$. Hence $q \in T(\hat{\alpha}_1 \cdots \hat{\alpha}_n)$ and $\epsilon E(x)\, q(x) > 0$ for every $x \in M^+ \cup M^-$.

One can argue in the same way in the case of restricted range approximation. So we obtain uniqueness results as in Taylor [12] and Loeb *et al.* [8].

(6) *Linear Approximation with Interpolatory Constraints*

Suppose $U = \operatorname{span}\{u_1 \cdots u_n\}$ is a Haar space on X. Let $m \in \mathbb{N}$, $m < n$, $0 \leqslant x_1 < \cdots < x_m \leqslant 1$ and $y_1 \cdots y_m \in \mathbb{R}$. $p: E \to \mathbb{R}^m$ is given by $(\alpha_1 \cdots \alpha_n) \mapsto (\Sigma \alpha_i u_i(x_1) - y_1, \ldots, \Sigma \alpha_i u_i(x_m) - y_m)$. Then p is linear and surjective, thus $P = \{(\alpha_1 \cdots \alpha_n) \mid p(\alpha_1 \cdots \alpha_n) = \Theta\}$ is regular and (F, P) has property R.

If $v_0 \in F[P]$ and $\hat{M}$ is a T-signature with $\{x_1 \cdots x_m\} \cap (M^+ \cup M^-) = \varnothing$ then (F, P) has the property TU at $(v_0, \hat{M})$. For $v \in F[P]$, $v \neq v_0$ and $\hat{M}_1 < \hat{M}$ with $Z(v - v_0) \supset M_1^+ \cup M_1^-$. $v = \Sigma \alpha_j u_j$. Then $T(\alpha_1 \cdots \alpha_n) = \{q \in U \mid q(x_j) = 0, j = 1, 2, \ldots, m\}$. Since $M_1^+ \cup M_1^-$ contains at most $n - m - 1$ points there is a $q \in U$ with $q(x) = 1$ if $x \in M_1^+$, $q(x) = -1$ if $x \in M^-$ and $q(x_j) = 0, j = 1, 2, \ldots, m$. Then $q \in T(\alpha_1 \cdots \alpha_n)$ and $\epsilon_{\hat{M}_1}(x)\, q(x) > 0$ for every $x \in M_1^+ \cup M_1^-$.

If $w \in C[0, 1]$, $w(x_j) = y_j$ $(j = 1, 2,..., m)$, then $(M^{=}_{w-v_0} \cup M^{-}_{w-v_0}) \cap \{x_1 \cdots x_m\} = \varnothing$ if $w \neq v_0$. Then (F, P) has the property TU at $(v_0, \hat{M}_{w-v_0})$. We can apply Theorem 6 to obtain results as in Deutsch [4] and Warth [14].

(7) *Linear Approximation with Parameter Constraints*

Let $E = \mathbb{R}^{n+1}$, $X = [0, 1]$, $u_0,..., u_n \in C(X)$ linear independent and let $F: E \to C(X)$ be given by $(\alpha_0,..., \alpha_n) \mapsto \Sigma \alpha_i u_i$. Instead of $F(\alpha)$ let us write F_α. Let $0 \leqslant k_1 < \cdots < k_r \leqslant n$ and $0 \leqslant l_1 < \cdots < l_s \leqslant n$ so that $\{k_1,..., k_r\} \cap \{l_1,..., l_s\} = \varnothing$.

Let $A, B \subset \{1, 2,..., r\}$ and for every $j \in A$, $a_j \in \mathbb{R}$, for every $j \in B$, $b_j \in \mathbb{R}$ and $c_1 \cdots c_s \in \mathbb{R}$. Let

$$P = \{(\alpha_0 \cdots \alpha_n) \in E \mid a_j \leqslant \alpha_{k_j} \text{ for every } j \in A,\ \alpha_{k_j} \leqslant b_j \text{ for every } j \in B,\ \alpha_{l_j} = c_j,\ i = 1, 2,..., s\}.$$

Suppose $a_j < b_j$ if $j \in A \cap B$. If $\alpha \in P$ then

$$T(\alpha) = \left\{ \sum_{i=0}^{n} \beta_i u_i \,\middle|\, \beta_{k_j} \geqslant 0 \text{ if } \alpha_{k_j} = a_j \text{ and } j \in A,\ \beta_{k_j} \leqslant 0 \text{ if } \alpha_{k_j} = b_j \text{ and } j \in B,\ \beta_{l_j} = 0, j = 1, 2,..., s \right\}.$$

Then P is regular and (F, P) has property R. $F[P]$ is convex, hence an α-sun.

Now suppose $u_i(x) = x^i$ for $x \in [0, 1]$ and $i = 0, 1,..., n$. Then

$$F[P] = \left\{ F_\alpha = \sum_{i=0}^{n} \alpha_i u_i \,\middle|\, \bar{a}_j \leqslant F_\alpha^{(k_j)}(0) \text{ for every } j \in A,\ F_\alpha^{(k_j)}(0) \leqslant \bar{b}_j \text{ for every } j \in B,\ F_\alpha^{(l_j)}(0) = \bar{c}_j, j = 1, 2,..., s \right\}$$

and for every $\alpha \in P$,

$$T(\alpha) = \left\{ q = \sum_{i=0}^{n} \beta_i u_i \,\middle|\, q^{(k_j)}(0) \geqslant 0 \text{ if } F_\alpha^{(k_j)}(0) = \bar{a}_j \text{ and } j \in A,\ q^{(k_j)}(0) \leqslant 0 \text{ if } F_\alpha^{(k_j)}(0) = \bar{b}_j \text{ and } j \in B,\ q^{(l_j)}(0) = 0, j = 1, 2,..., s \right\}$$

with $\bar{a}_j = k_j!\, a_j$ $(j \in A)$, $\bar{b}_j = k_j!\, b_j$ $(j \in B)$, $\bar{c}_j = l_j!\, c_j$ $(j = 1, 2,..., s)$. If $q^{(j)}(0) = 0$ for $q \in \operatorname{span}\{u_0 \cdots u_n\}$ and $j \in \{0, 1,..., n\}$ then $q \in \operatorname{span}(\{u_0 \cdots u_n\} \setminus \{u_j\})$.

Thus

$$\operatorname{span}(\{u_0 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\alpha) \cup I_2(\alpha)}\}) \subset T(\alpha),$$

where

$$I_1(\alpha) = \{j \in A \mid F_\alpha^{(k_j)}(0) = \bar{a}_j\},$$

$$I_2(\alpha) = \{j \in B \mid F_\alpha^{(k_j)}(0) = \bar{b}_j\}.$$

Let $r_1(\alpha)$ (resp. $r_2(\alpha)$) be the number of elements of $I_1(\alpha)$ (resp. $I_2(\alpha)$). We note that $\operatorname{span}\{u_1 \cdots u_r\}$ is a Haar space on $(0, \gamma]$ for every $\gamma > 0$ and every collection of numbers $\mu_1 \cdots \mu_r \in \mathbb{N}_0$ with $0 \leqslant \mu_1 < \cdots < \mu_r \leqslant n$ (see Karlin and Studden [5]).

Now assume $k_1 > 0$, $l_1 > 0$. Then $r_1(\alpha) + r_2(\alpha) + s < n + 1$ for every $\alpha \in P$. Let $m(\alpha) = (n + 1) - (r_1(\alpha) + r_2(\alpha) + s)$ $(\alpha \in P)$. Suppose v_0, $v_1 \in F[P]$ and $v_0 \neq v_1$. Let $\hat{\alpha} \in P$ so that $F(\hat{\alpha}) = \hat{v} = \frac{1}{2}(v_1 + v_0)$. Then $\hat{v} - v_0$ has at most $m(\hat{\alpha}) - 1$ zeros in $[0, 1]$. For $j \in I_1(\hat{\alpha})$, then $F_\alpha^{(k_j)}(0) = \bar{a}_j$ which implies $v_1^{(k_j)}(0) - v_0^{(k_j)}(0) = 0$ and $\hat{v}^{(k_j)}(0) - v_0^{(k_j)}(0) = 0$. Analogously $j \in I_2(\hat{\alpha})$ implies $\hat{v}^{(k_j)}(0) - v_0^{(k_j)}(0) = 0$. Furthermore $v^{(l_j)}(0) - v_0^{(l_j)}(0) = 0$. Thus $\hat{v} - v_0 \in \operatorname{span}(\{u_0 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\hat{\alpha}) \cup I_2(\hat{\alpha})}\})$ which is a Haar space on $(0, 1]$, so $\hat{v} - v_0$ has at most $m(\alpha) - 1$ zeros in $(0, 1]$. Suppose $\hat{v}(0) - v_0(0) = 0$. Then $\hat{v} - v_0 \in \operatorname{span}(\{u_1 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\hat{\alpha}) \cup I_2(\hat{\alpha})}\}) \neq \{\Theta\}$ hence $\hat{v} - v_0$ has at most $m(\hat{\alpha}) - 2$ $(\geqslant 0)$ zeros in $(0, 1]$.

Let us now show that (F, P) has the property TU. Let $v_0 \in F[P]$ and $\hat{M}$ be a T-signature. Suppose there is a $v_1 \in F[P]$, $v_1 \neq v_0$ and $\hat{M}_1 < \hat{M}$ with $Z(v_1 - v_0) \supset M_1^+ \cup M_1^-$.

Let $\hat{v} = \frac{1}{2}(v_1 + v_0) = F(\hat{\alpha})$. Then $F(\hat{\alpha}) \neq v_0$, $Z(F(\hat{\alpha}) - v_0) \supset M_1^+ \cup M_1^-$ and $M_1^+ \cup M_1^-$ contains at most $m(\hat{\alpha}) - 1$ $(\geqslant 1)$ points.

Suppose $0 \notin M_1^+ \cup M_1^-$. Let $q \in \operatorname{span}(\{u_0 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\hat{\alpha}) \cup I_2(\hat{\alpha})}\})$ with $q(x) = \epsilon_{\hat{M}}(x)$ for every $x \in M_1^+ \cup M_1^-$. Then $q \in T(\hat{\alpha})$.

If $0 \in M_1^+ \cup M_1^-$ then let $q_1 \in \operatorname{span}(\{u_1 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\hat{\alpha}) \cup I_2(\hat{\alpha})}\})$ with $q_1(x) = \epsilon_{\hat{M}}(x)$ for every $x \in M_1^+ \cup M_1^- \backslash \{0\}$. Let $q_\lambda = q_1 + \lambda \epsilon_{\hat{M}}(0)\, u_0 \in \operatorname{span}(\{u_0 \cdots u_n\} \backslash \{u_{l_1} \cdots u_{l_s}, (u_{k_j})_{j \in I_1(\hat{\alpha}) \cup I_2(\hat{\alpha})}\})$. If λ is sufficiently small then $q_\lambda(x)\, \epsilon_{\hat{M}}(x) > 0$ for every $x \in M_1^+ \cup M_1^-$. Furthermore $q_\lambda \in T(\hat{\alpha})$.

So we have proved that (F, P) has property TU at $(v_0, \hat{M})$ (for every $v_0 \in F[P]$ and every T-signature M). By Theorem 7 every $w \in C[0, 1]$ has at most one best approximation. This has been proved by a different method by Roulier and Taylor [11].

4. Final Remark

A comparison of this paper with Warth [15] shows that L_1-approximation and uniform approximation of continuous functions can be treated by similar theories.

REFERENCES

1. D. BRAESS, Kritische Punkte bei der nicht-linearen Tschebyscheff-Approximation, *Math. Z.* **132** (1973).
2. B. BROSOWSKI, Über die Eindeutigkeit der asymptotischkonvexen Tschebyscheff-Approximation, ISNM 7 (1967).
3. B. BROSOWSKI AND R. WEGMANN, Charakterisierung bester Approximationen in normierten Vektorräumen, *J. Approximation Theory* **3** (1970), 369–397.
4. F. DEUTSCH, Uniform approximation with interpolatory constraints, *J. Math. Anal. Appl.* **24** (1968), 62–79.
5. S. KARLIN AND W. S. STUDDEN, "Tschebyscheff Systems: With Applications in Statistics and Analysis," New York, 1966.
6. A. KIRSCH, W. WARTH, AND J. WERNER, Notwendige Optimalitätsbedingungen und Anwendungen, Lecture Note, Universität Göttingen, 1977; Lecture Notes in Economics and Mathematical Systems, Vol. 152, Springer-Verlag, Berlin/New York.
7. W. KRABS, Über differenzierbare asymptotisch-konvexe Funktionenfamilien bei der nicht-linearen gleichmässigen Approximation, *Arch. Rational Mech. Anal.* **27** (1967).
8. H. L. LOEB, D. G. MOURSUND, AND G. D. TAYLOR, Uniform rational weighted approximation having restricted ranges, *J. Approximation Theory* **1** (1968), 401–411.
9. G. MEINARDUS AND D. SCHWEDT, Nichtlineare Approximation, *Arch. Rational Mech. Anal.* **17** (1964).
10. I. R. RICE, The approximation of functions, Addison–Wesley, London, 1964.
11. I. A. ROULIER AND G. D. TAYLOR, Uniform approximation by polynomials having bounded coefficients, *Abh. Math. Sem. Univ. Hamburg* **36** (1971).
12. G. D. TAYLOR, On approximation by polynomials having restricted ranges, *SIAM J. Numer. Anal.* **5** (1968).
13. W. WARTH, Approximation with constraints in normed linear spaces, *J. Approximation Theory*, in press.
14. W. WARTH, Approximation with interpolatory constraints, *J. Math. Anal. Appl.* **64** (1978), 488–493.
15. W. WARTH, On a theorem of Jackson concerning best approximations, submitted.